

Hopf instantons, Chern-Simons vortices and Heisenberg ferromagnets

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Abstract. *The dimensional reduction of the three-dimensional model (related to Hopf maps) of Adam et al. is shown to be equivalent to (i) either the static, fixed-chirality sector of the non-relativistic spinor-Chern-Simons model in $2 + 1$ dimensions, (ii) or a particular Heisenberg ferromagnet in the plane.*

1. Scalar Chern-Simons vortices and Hopf instantons

In the non-relativistic Chern-Simons model of Jackiw and Pi [1], one considers a scalar field Φ which satisfies a second-order non-linear Schrödinger equation,

$$(1.1) \quad iD_t\Psi = \frac{D_i D^i}{2m}\Psi - g|\Psi|^2\Psi = 0,$$

while the dynamics of the gauge field is governed by the Chern-Simons field/current identities. When the coupling constant g is minus or plus the inverse of the Chern-Simons coupling constant κ , static solutions arise by solving instead the self-duality equations,

$$(1.2) \quad D_{\pm}\Psi \equiv (D_1 \pm D_2)\Psi = 0, \quad (D_k = \partial_k - iA_k),$$

supplemented with one of the Chern-Simons equations, namely

$$(1.3) \quad \kappa B \equiv \kappa \epsilon^{ij} \partial_i A^j = -\varrho,$$

where $\varrho = \Phi^* \Phi$ is the particle density. Expressing the gauge potential from (1.2) one finds that the other Chern-Simons equations, $\kappa E^i \equiv -\kappa(\partial_i A^0 + \partial_t A^i) = \epsilon^{ij} J^j$, merely fixes A_t . Then, inserting into (1.3) yields the Liouville equation, whose well-known solutions provide us with Chern-Simons vortices which carry electric and magnetic fields. The self-dual solutions represent furthermore the absolute minima of the energy, cf. [1].

In a recent paper, Adams, Muratori and Nash [2] consider instead a massless two-spinor $\Phi = \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}$ on ordinary 3-space, coupled to a (euclidean) Chern-Simons field. Their field equations read

$$(1.4) \quad D_i \sigma_i \Phi = 0,$$

$$(1.5) \quad \Phi^\dagger \sigma_i \Phi = B_i.$$

Note that this model only contains a (three-dimensional) magnetic but no electric field. These authors also mention that assuming independence of x_3 and setting $A_3 = 0$, their model will reduce to the planar self-dual Jackiw-Pi system, (1.2-3). The third component of (1.5) requires in fact

$$(1.6) \quad |\Phi_+|^2 - |\Phi_-|^2 = B;$$

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the two other components imply, however, that either Φ_+ or Φ_- has to vanish. Therefore, the reduced equations read finally one or the other of

$$(1.7) \quad D_{\pm}\Phi_{\mp} = 0, \quad B = \pm|\Phi_{\pm}|^2, \quad \text{and} \quad \Phi_{\mp} = 0.$$

Fixing up the sign problem by including a Chern-Simons coupling constant κ , these equations look indeed *formally* the same as in the self-dual Jackiw-Pi case. They have, however, a slightly different interpretation: they are purely magnetic, while those of Jackiw and Pi have a non-vanishing electric field. Let us underline that the equations (1.7) differ from the second-order field equation (1.1).

2. Spinor vortices

Here we point out that the model of Adam et al. reduces rather more naturally to a particular case of our spinor model in $2+1$ dimensions [3]. In this theory, the 4-component Dirac spinor with components Φ_- , χ_- , χ_+ and Φ_+ satisfies the Lévy-Leblond equations [4]

$$(2.1) \quad \begin{cases} (\vec{\sigma} \cdot \vec{D})\Phi + 2m\chi = 0, \\ D_t\Phi + i(\vec{\sigma} \cdot \vec{D})\chi = 0, \end{cases}$$

where Φ and χ are two-component ‘Pauli’ spinors $\Phi = \begin{pmatrix} \Phi_- \\ \Phi_+ \end{pmatrix}$ and $\chi = \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix}$. This non-relativistic Dirac-type equation is completed with the Chern-Simons equations

$$(2.2) \quad \begin{aligned} B &= (-1/\kappa)(|\Phi_+|^2 + |\Phi_-|^2), \\ E_i &= (1/\kappa)\epsilon_{ij}J_j, \quad J_j = i(\Phi^\dagger\sigma_j\chi - \chi^\dagger\sigma_j\Phi). \end{aligned}$$

In the static and purely magnetic case, $A_t = 0$, and choosing $\chi_+ = \chi_- = 0$, the second equation in (2.2) is identically satisfied, leaving us with the coupled system

$$(2.3) \quad \begin{cases} D_+\Phi_- = 0, \\ D_-\Phi_+ = 0, \\ B = (-1/\kappa)(|\Phi_+|^2 + |\Phi_-|^2). \end{cases}$$

Choosing a fixed chirality, $\Phi_- \equiv 0$ or $\Phi_+ \equiv 0$, yields furthermore either of the two systems

$$(2.4) \quad \begin{aligned} D_{\pm}\Phi_{\mp} &= 0, \\ B &= (-1/\kappa)|\Phi_{\pm}|^2, \end{aligned}$$

which, for $\kappa = 1$, are precisely (1.7). For both signs, the equations (2.4) reduce to the Liouville equation; regular solutions were obtained for Φ_+ when $\kappa < 0$, and for Φ_- when $\kappa > 0$. They are again purely magnetic, and carry non-zero spin.

It would be easy keep both terms in (1.7) by allowing a non-vanishing (but still x_3 -independent) A_3 . Then one would loose the equations $D_{\pm}\Phi_{\mp} = 0$, however. The impossibility to having both components in (2.3) but not in (1.7) comes from the type of reduction performed: while for spinors one eliminates non-relativistic time, (1.7) comes from a spacelike reduction. The difference is also related to the structure of the Lévy-Leblond equation (2.1), which can be obtained by *lightlike* reduction from a massless Dirac equation in 4-dimensions, while (1.4) comes by *spacelike* reduction [3].

It is interesting to observe that eliminating χ in favor of Φ in the Lévy-Leblond equation (2.3) yields

$$(2.5) \quad iD_t\Phi = \left[-\frac{1}{2m}D_iD^i + \frac{1}{2m\kappa}(|\Phi_+|^2 + |\Phi_-|^2)\sigma_3 \right]\Phi.$$

For both chiralities, we get hence a second-order equation of the Jackiw-Pi form (1.1), but with opposite signs i.e., with attractive/repulsive coupling.

It is worth noting that minima of the energy correspond to the coupled equations (2.3) and *not* to (2.4). In fact, the identity

$$|\vec{D}\Phi|^2 = |D_+\Phi_-|^2 + |D_-\Phi_+|^2 - (1/2m\kappa)|\Phi|^2\Phi^\dagger\sigma_3\Phi + \text{surface terms}$$

shows that the energy of a field configuration,

$$H = \int \left\{ (1/2m)|\vec{D}\Phi|^2 + (1/2m\kappa)|\Phi|^2\Phi^\dagger\sigma_3\Phi \right\} d^2\vec{x},$$

is actually

$$(2.6) \quad H = \frac{1}{2m} \int d^2\vec{r} \{ |D_+\Phi_-|^2 + |D_-\Phi_+|^2 \},$$

which is positive definite, $H \geq 0$, provided the currents vanish at infinity. The “Bogomolny” bound is furthermore saturated precisely when (2.3) holds. Its solutions are therefore stable; (2.3) should be considered as the true self-duality condition.

3. Heisenberg ferromagnets

The relative minus sign of the component densities in the “provisional” formula (1.6) differs from ours in (2.3), and is rather that in the 2-dimensional Heisenberg model studied by Martina et al. [5]. Here the spin, represented by a unit vector \mathbf{S} , satisfies the Landau-Lifschitz equation $\partial_t \mathbf{S} = \mathbf{S} \times \Delta \mathbf{S}$. In the so-called tangent-space representation, \mathbf{S} is replaced by two complex fields, Ψ_+ and Ψ_- , each of which satisfies a (second-order) non-linear Schrödinger equation,

$$(3.1) \quad iD_t\Psi_\pm = -\left[D_i D^i + 8|\Psi_\pm|^2\right]\Psi_\pm,$$

as well as a geometric constraint, $D_+\Psi_- = D_-\Psi_+$. The covariant derivatives here refer to a Chern-Simons-type abelian gauge field,

$$(3.2) \quad \begin{aligned} B &= -8(|\Psi_+|^2 - |\Psi_-|^2), \\ E_i &= 8\epsilon_{ij}J_j, \quad J_i = (\Psi_+^* D_i \Psi_+ - \Psi_+ (D_i \Psi_+)^*) - (\Psi_-^* D_i \Psi_- - \Psi_- (D_i \Psi_-)^*). \end{aligned}$$

It is now easy to check that in the static and purely magnetic case, these equations can be solved by the first-order coupled system

$$(3.3) \quad \begin{aligned} D_\pm \Psi_\mp &= 0, \\ B &= -8(|\Psi_+|^2 - |\Psi_-|^2). \end{aligned}$$

For $\Psi_+ = 0$ or $\Psi_- = 0$, we get once again the equation of Adams et al.. In the general case, (3.3) leads to an interesting generalization of the Liouville equation : making use of its conformal properties, Martina et al. have shown that it can be transformed into the “sinh-Gordon” form

$$(3.4) \quad \Delta\sigma = -\sinh\sigma,$$

where σ is suitably defined from Ψ_+ and Ψ_- . Although this equation has no finite-energy regular solution defined over the whole plane [6], it admits doubly-periodic solutions i. e. solutions defined in cells with

periodic boundary conditions on the boundary [7]. This generalises the results of Olesen [8] in the scalar case. A similar calculation applied to the general SD equations, (2.3), of our spinor model would yield

$$(3.5) \quad \Delta\sigma = -\cosh\sigma,$$

whose (doubly periodic) solutions could be interpreted as non-linear superpositions of the chiral vortices in [DHP].

References

- [1] R. Jackiw and S-Y. Pi, Phys. Rev. Lett. **64**, 2969 (1990); Phys. Rev. **D42**, 3500 (1990); For reviews, see, e. g., R. Jackiw and S-Y. Pi, Prog. Theor. Phys. Suppl. **107**, 1 (1992) or G. Dunne, *Self-Dual Chern-Simons solitons*. Springer Lecture Notes in Physics (New Series) **36**, (1995).
- [2] C. Adam, B. Muratori, and C. Nash, Phys. Lett. **B 479**, 329 (2000).
- [3] C. Duval, P. A. Horváthy and L. Palla, Phys. Rev. **D52**, 4700 (1995); Ann. Phys. (N.Y.) **249**, 265 (1996). The same self-dual equations arise in the relativistic model of Y. M. Cho, J. W. Kim, and D. H. Park, Phys. Rev. **D45**, 3802 (1992).
- [4] J-M. Lévy-Leblond, Comm. Math. Phys. **6**, 286 (1967).
- [5] L. Martina, O. K. Pashaev, G. Soliani, Phys. Rev. **B48**, 15787 (1993).
- [6] G. Dunne, R. Jackiw, S.-Y. Pi, Trugenberger, Phys. Rev. **D43**, 1332 (1991).
- [7] A. C. Ting, H. H. Chen and Y. C. Lee, Phys. Rev. Lett. **53**, 1348 (1984); Physica **26D**, 37 (1987).
- [8] P. Olesen, Phys. Lett. **B265**, 361 (1991); *ibid.* **B268**, 389 (1991).